Beach, FL; Jahangeer Kholdi and Farideh Firoozbakht, University of Isfahan, Khansar, Iran; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Adrian Naco, Polytechnic University, Tirana, Albania; Perfetti Paolo, Department of Mathematics, "Tor Vergata" University, Rome, Italy, and the proposer.

- **5281:** *Proposed by Arkady Alt, San Jose, CA*

For the sequence $\{a_n\}_{n\geq 1}$ defined recursively by $a_{n+1} = \dfrac{a_n}{1+a_n^p}$ for $n \in \mathcal{N}, a_1 = a > 0$, determine all positive real p for which the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

Solution 1 by Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy

Answer: $p < 1$.

Proof: Since $a_{n+1} < a_n$, $a_n \to 0$.
It follows that

$$a_{n+1} = a_n - a_n^{p+1} + a_n^{2p+1} + O(a_n^{3p+1})$$

We employ the standard result of the exercise num.174 at page 38 of the book by G. Pólya, G. Szegö, *Problems and Theorems in Analysis, I.*

Assume that $0 < f(x) < x$ and $f(x) = x - ax^k + bx^l + x^l\varepsilon(x)$, $\lim_{x\to 0}\varepsilon(x) = 0$, for $0 < x < x_0$ where $1 < k < l$ and a, b both positive. The sequence x_n defined by $x_{n+1} = f(x_n)$ satisfies

$$\lim_{n\to\infty} n^{1/(k-1)} x_n = (a(k-1))^{-1/(k-1)}.$$

In our case we have $a = 1$, $k = p+1$, $b = 1$, $l = 2p+1$. Thus the sequence satisfies

$$a_n = p^{-1/p} n^{-1/p} + o(n^{-1/p})$$

and then the series converges if and only if $p < 1$.

Solution 2 by Kee-Wai Lau, Hong Kong, China

We show that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent if $0 < p < 1$ and divergent if ≥ 1.

We assume in what follows that $n \in N$. Clearly $a_n > 0$ and by the given recursive relation, we have $a_{n+1} < a_n$. Therefore $L = \lim\limits_{n\to\infty} a_n$ exists and from $L = \dfrac{L}{1+L^p}$, we see that $L = 0$. Inductively, we have

$$a_{n+1} = \frac{a}{\prod_{k=1}^{n}(1+a_k^p)}. \qquad (1)$$

By making use of the well-known inequality $1 + x < e^x$ for $x > 0$, we deduce from (1) that $a_{n+1} > ae^{-\sum_{k=1}^{n} a_k^p} > 0$. Since $\lim\limits_{n\to\infty} a_{n+2} = 0$, so $\displaystyle\sum_{k=1}^{n} a_k^p$ is divergent. Now there exits $k_0 \in N$, depending at most on a and p, such that $a_k < 1$ whenever $k > k_0$. Hence if $p \geq 1$, then for any integer $M > k_0$, we have $\sum_{k=k_0+1}^{M} a_k \geq \sum_{k=k_0+1}^{M} a_k^p$. Thus $\sum_{k=+1}^{\infty} a_k$ is divergent.

We next consider the case $0 < p < 1$. Let $m = \left\lfloor \frac{1}{1-p} \right\rfloor + 1$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x. By (1), for any $n > m$, we have

$$0 < a_{n+1} \leq \frac{a}{(1+a_n^p)^n} < \frac{a}{(1+a_{n+1}^p)^n} < \frac{a}{\binom{n}{m} a_{n+1}^{mp}},$$

so that

$$0 < a_{n+1} < \left(\frac{am!}{\prod_{k=0}^{m-1}(n-k)}\right)^{1/(1+mp)} \leq \left(\frac{am!}{(n-m+1)^m}\right)^{1/(1+mp)}.$$

It is easy to check that $\frac{m}{1+mp} > 1$, and so $\sum_{n=1}^{\infty} a_n$ is convergent.

This completes the solution.

Also solved by Ed Gray, Highland Beach, FL, and the proposer.

- **5282:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate

$$\int_0^1 x \ln\left(\sqrt{1+x} - \sqrt{1-x}\right) \ln\left(\sqrt{1+x} + \sqrt{1-x}\right) dx.$$

Solution 1 by Anastasios Kotronis, Athens, Greece

Using the identity

$$ab = \frac{1}{4} \cdot a + b^2 - a - b^2,$$

with $a = \ln\sqrt{1+x} - \sqrt{1-x}$ and $b = \ln\sqrt{1+x} + \sqrt{1-x}$ we have

$$\begin{aligned}
I &= \int_0^1 x \ln\sqrt{1+x} - \sqrt{1-x} \ln\sqrt{1+x} + \sqrt{1-x}\, dx \\
&= \frac{1}{4}\int_0^1 x\ln^2(2x) - \ln^2 \frac{1-\sqrt{\frac{1-x}{1+x}}}{1+\sqrt{\frac{1-x}{1+x}}}\, dx \\
&= \frac{1}{4}\int_0^1 x\ln^2(2x)\, dx - \frac{1}{4}\int_0^1 x\ln^2 \frac{1-\sqrt{\frac{1-x}{1+x}}}{1+\sqrt{\frac{1-x}{1+x}}}\, dx \\
&= I_1 - I_2.
\end{aligned}$$